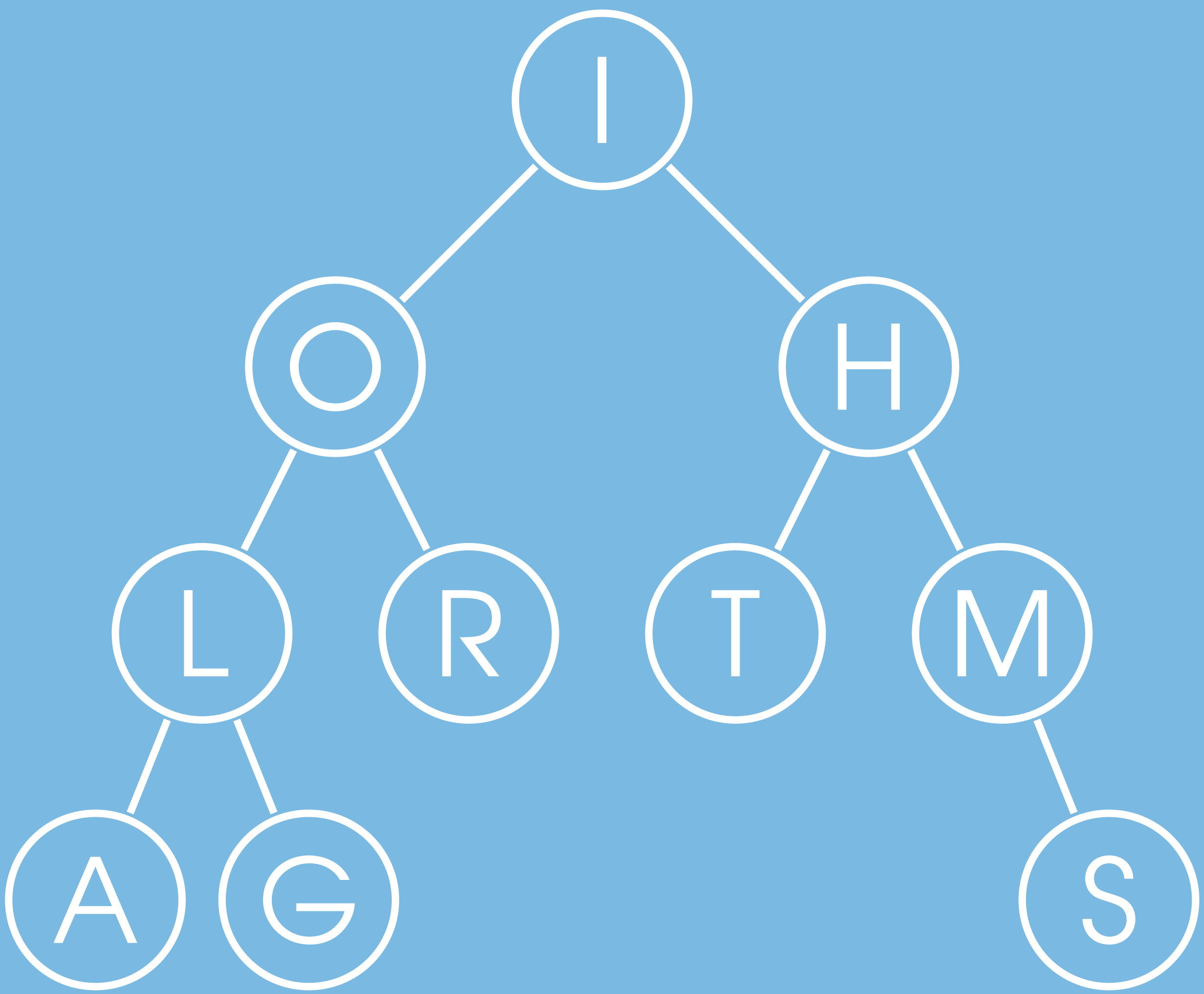
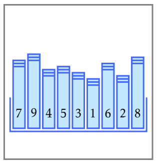
LEARNING ALGORITHMS THROUGH PROGRAMMING AND PUZZLE SOLVING

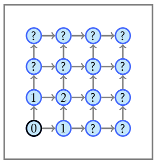


by Alexander Kulikov and Pavel Pevzner

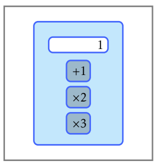
Contents



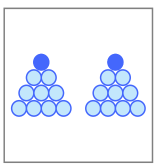
Book Sorting. Rearrange books on the shelf (in the increasing order of heights) using minimum number of swaps.



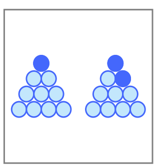
Number of Paths. Find out how many paths are there to get from the bottom left circle to any other circle and place this number inside the corresponding circle.



Antique Calculator. Find the minimum number of operations needed to get a positive integer n from the integer 1 using only three operations: add 1, multiply by 2, or multiply by 3.

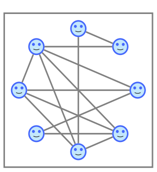


Two Rocks Game. There are two piles of ten rocks. In each turn, you and your opponent may either take one rock from a single pile, or one rock from both piles. Your opponent moves first and the player that takes the last rock wins the game. Design a winning strategy.

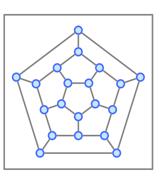


Three Rocks Game. There are two piles of ten rocks. In each turn, you and your opponent may take up to three rocks. Your opponent moves first and the player that takes the last rock wins the game. Design a winning strategy.

Map Coloring. Use minimum number of colors such that neighboring countries are assigned different colors and each country is assigned a single color.



Clique Finding. Find the largest group of mutual friends (each pair of friends is represented by an edge).



Icosian Game. Find a cycle visiting each node exactly once.

FASTROCKS(n, m):
if n and m are both even:
    return L
else:
    return W

However, though FASTROCKS is more efficient than ROCKS, it may be difficult to modify it for similar games, for example, a game in which each player can move up to three rocks at a time from the piles. This is one example where the slower algorithm is more instructive than a faster one.

Exercise Break. Play the Three Rocks game using our interactive puzzle and construct the dynamic programming table similar to the table above for this game.

2.5 Recursive Algorithms

Recursion is one of the most ubiquitous algorithmic concepts. Simply, an algorithm is recursive if it calls itself.

The *Towers of Hanoi puzzle* consists of three pegs, which we label from left to right as 1, 2, and 3, and a number of disks of decreasing radius, each with a hole in the center. The disks are initially stacked on the left peg (peg 1) so that smaller disks are on top of larger ones. The game is played by moving one disk at a time between pegs. You are only allowed to place smaller disks on top of larger ones, and any disk may go onto an empty peg. The puzzle is solved when all of the disks have been moved from peg 1 to peg 3. Try our interactive puzzle Hanoi Towers to figure out how to move all disks from one peg to another.

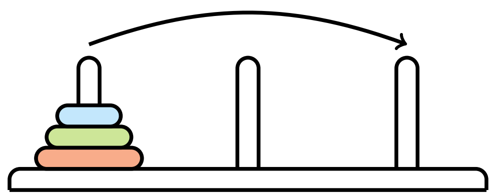
Towers of Hanoi Problem
Output a list of moves that solves the Towers of Hanoi.

Input: An integer n.
Output: A sequence of moves that solve the n-disk Towers of Hanoi puzzle.

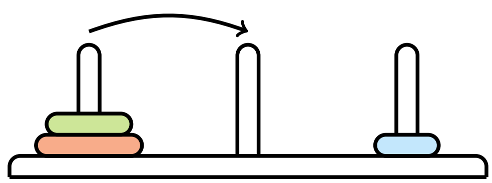
Solving the puzzle with one disk is easy: move the disk to the right peg. The two-disk puzzle is not much harder: move the small disk to the middle peg, then the large disk to the right peg, then the small disk to the right peg to rest on top of the large disk.

The three-disk puzzle is somewhat harder, but the following sequence of seven moves solves it:

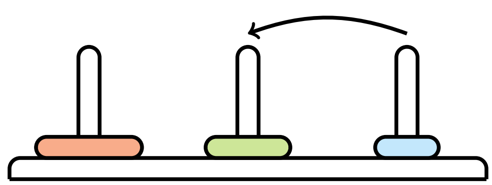
1. Move disk from peg 1 to peg 3



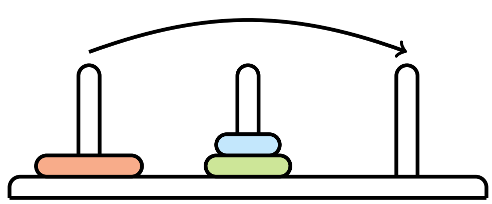
2. Move disk from peg 1 to peg 2



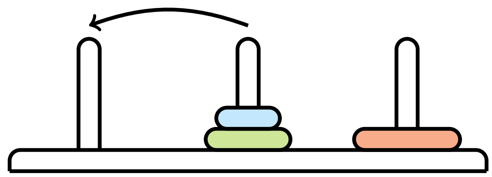
3. Move disk from peg 3 to peg 2



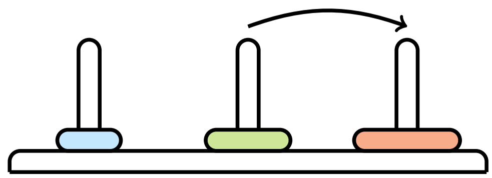
4. Move disk from peg 1 to peg 3



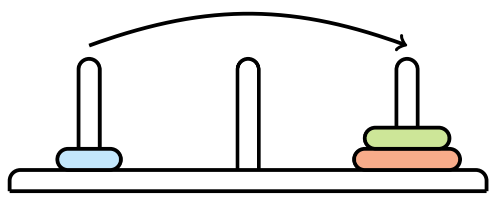
5. Move disk from peg 2 to peg 1



6. Move disk from peg 2 to peg 3



7. Move disk from peg 1 to peg 3



Now we will figure out how many steps are required to solve a four-disk puzzle. You cannot complete this game without moving the largest disk. However, in order to move the largest disk, we first had to move all the smaller disks to an empty peg. If we had four disks instead of three, then we would first have to move the top three to an empty peg (7 moves), then move the largest disk (1 move), then again move the three disks from their temporary peg to rest on top of the largest disk (another 7 moves). The whole procedure will take $7 + 1 + 7 = 15$ moves.

More generally, to move a stack of size n from the left to the right peg, you first need to move a stack of size $n-1$ from the left to the middle peg, and then from the middle peg to the right peg once you have moved the n-th disk to the right peg. To move a stack of size $n-1$ from the middle to the right, you first need to move a stack of size $n-2$ from the middle to the left, then move the $(n-1)$-th disk to the right, and then move the stack of size $n-2$ from the left to the right peg, and so on.

At first glance, the Towers of Hanoi Problem looks difficult. However, the following *recursive algorithm* solves the Towers of Hanoi Problem with just 9 lines!

```
HanoiTowers(n, fromPeg, toPeg)
if n = 1:
   output "Move disk from peg fromPeg to peg toPeg"
   return
unusedPeg ← 6 − fromPeg − toPeg
HanoiTowers(n − 1, fromPeg, unusedPeg)
output "Move disk from peg fromPeg to peg toPeg"
HanoiTowers(n − 1, unusedPeg, toPeg)
```

The variables *fromPeg*, *toPeg*, and *unusedPeg* refer to the three different pegs so that HanoiTowers($n, 1, 3$) moves n disks from the first peg to the third peg. The variable *unusedPeg* represents which of the three pegs can serve as a temporary destination for the first $n-1$ disks. Note that $fromPeg + toPeg + unusedPeg$ is always equal to $1+2+3=6$, so the value of the variable *unusedPeg* can be computed as $6 - fromPeg - toPeg$. Table below shows the result of $6 - fromPeg - toPeg$ for all possible values of *fromPeg* and *toPeg*.

fromPeg	*toPeg*	*unusedPeg*
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

After computing *unusedPeg* as $6 - fromPeg - toPeg$, the statements

```
HanoiTowers(n − 1, fromPeg, unusedPeg)
output "Move disk from peg fromPeg to peg toPeg"
HanoiTowers(n − 1, unusedPeg, toPeg)
```

solve the smaller problem of moving the stack of size $n-1$ first to the temporary space, moving the largest disk, and then moving the $n-1$ remaining disks to the final destination. Note that we do not have to specify which disk the player should move from *fromPeg* to *toPeg*: it is always the top disk currently residing on *fromPeg* that gets moved.

Although the Hanoi Tower solution can be expressed in just 9 lines of pseudocode, it requires a surprisingly long time to run. To solve a five-disk tower requires 31 moves, but to solve a hundred-disk tower would require more moves than there are atoms on Earth. The fast growth of the number of moves that HanoiTowers requires is easy to see by noticing that every time HanoiTowers$(n, 1, 3)$ is called, it calls itself twice for $n-1$, which in turn triggers four calls for $n-2$, and so on.

We can illustrate this situation in a *recursion tree*, which is shown in Figure 2.2. A call to HanoiTowers$(4, 1, 3)$ results in calls HanoiTowers$(3, 1, 2)$ and HanoiTowers$(3, 2, 3)$; each of these results in calls to HanoiTowers$(2, 1, 3)$, HanoiTowers$(2, 3, 2)$ and HanoiTowers$(2, 2, 1)$, HanoiTowers$(2, 1, 3)$, and so on. Each call to the subroutine HanoiTowers requires some amount of time, so we would like to know how much time the algorithm will take.

To calculate the running time of HanoiTowers of size n, we denote the number of disk moves that HanoiTowers(n) performs as $T(n)$ and notice that the following equation holds:

$$T(n) = 2 \cdot T(n-1) + 1\,.$$

Starting from $T(1) = 1$, this recurrence relation produces the sequence:

$$1, 3, 7, 15, 31, 63,$$

and so on. We can compute $T(n)$ by adding 1 to both sides and noticing

$$T(n) + 1 = 2 \cdot T(n-1) + 1 + 1 = 2 \cdot (T(n-1) + 1)\,.$$

If we introduce a new variable, $U(n) = T(n) + 1$, then $U(n) = 2 \cdot U(n-1)$. Thus, we have changed the problem to the following recurrence relation.

$$U(n) = 2 \cdot U(n-1)\,.$$

Starting from $U(1) = 2$, this gives rise to the sequence

$$2, 4, 8, 16, 32, 64, \ldots$$

implying that at $U(n) = 2^n$ and $T(n) = U(n) - 1 = 2^n - 1$. Thus, HanoiTowers$(n)$ is an exponential algorithm.

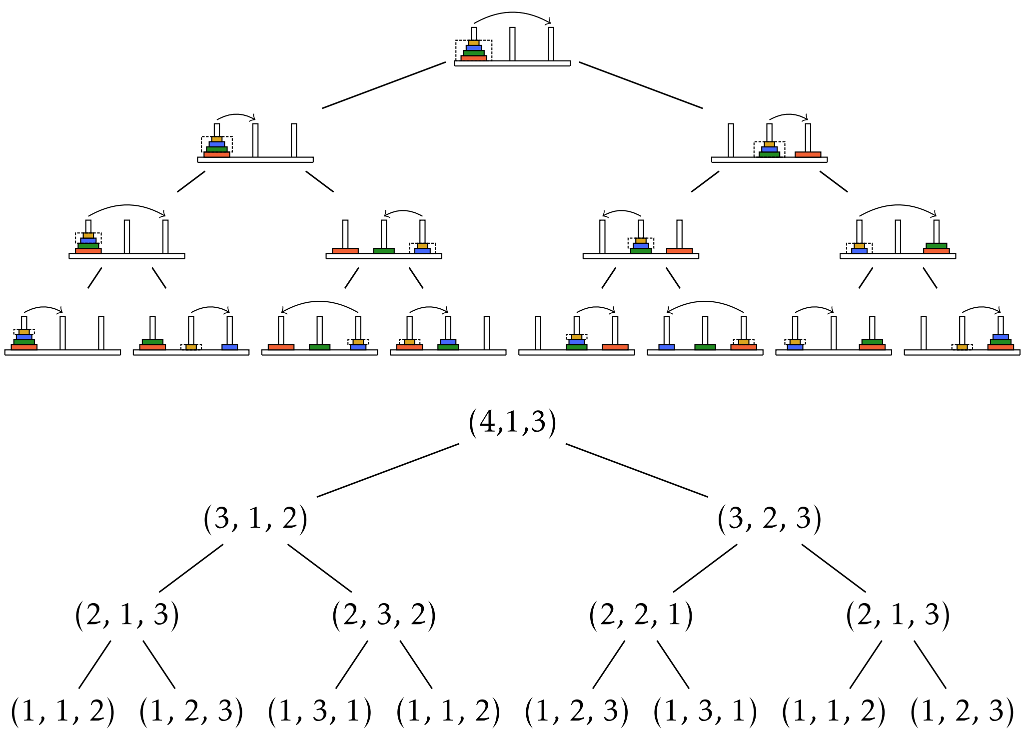
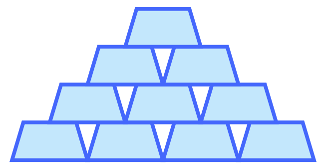


Figure 2.2: The recursion tree for a call to HanoiTowers$(4,1,3)$, which solves the Towers of Hanoi problem of size 4. At each point in the tree, (i,j,k) stands for HanoiTowers(i,j,k).

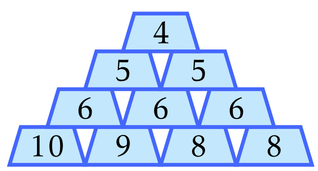
8.6 Maximum Amount of Gold

Maximum Amount of Gold Problem
Given a set of gold bars of various weights and a backpack that can hold at most W pounds, place as much gold as possible into the backpack.

> **Input:** A set of n gold bars of integer weights $w_1, \ldots, w_n$ and a backpack that can hold at most W pounds.
> **Output:** A subset of gold bars of maximum total weight not exceeding W.



You found a set of gold bars and your goal is to pack as much gold as possible into your backpack that has capacity W, i.e., it may hold at most W pounds. There is just one copy of each bar and for each bar you can either take it or not (you cannot take a fraction of a bar). Although all bars appear to be identical in the figure above, their weights vary as illustrated in the figure below.



A natural greedy strategy is to grab the heaviest bar that still fits into the remaining capacity of the backpack and iterate. For the set of bars shown above and a backpack of capacity 20, the greedy algorithm would select gold bars of weights 10 and 9. But an optimal solution, containing bars of weights 4, 6, and 10, has a larger weight!

Input format. The first line of the input contains an integer W (capacity of the backpack) and the number n of gold bars. The next line contains n integers $w_1, \ldots, w_n$ defining the weights of the gold bars.

Output format. The maximum weight of gold bars that fits into a backpack of capacity W.

Constraints. $1 \leq W \leq 10^4$; $1 \leq n \leq 300$; $0 \leq w_1, \dots, w_n \leq 10^5$.

Sample.

Input:

```
10 3
1 4 8
```

Output:

```
9
```

The sum of the weights of the first and the last bar is equal to 9.

Solution 1: Analyzing the Structure of a Solution

Instead of solving the original problem, we will check whether it is possible to fully pack our backpack with the gold bars: given n gold bars of weights $w_0, \dots, w_{n-1}$ (we switched to the 0-based indexing) and an integer W, is it possible to select a subset of them of the total weight W?

Exercise Break. Show how to use the solutions to this problem to solve the Maximum Amount of Gold Problem.

Assume that it is possible to fully pack the backpack: there exists a set $S \subseteq \{w_0, \dots, w_{n-1}\}$ of total weight W. Does it include the last bar of weight w_{n-1}?

Case 1: If $w_{n-1} \notin S$, then a backpack of capacity W can be fully packed using the first $n-1$ bars.

Case 2: If $w_{n-1} \in S$, then we can remove the bar of weight w_{n-1} from the backpack and the remaining bars will have weight $W - w_{n-1}$. Therefore, a backpack of capacity $W - w_{n-1}$ can be fully packed with the first $n-1$ gold bars.

In both cases, we reduced the problem to essentially the same problem with smaller number of items and possibly smaller backpack capacity. We thus consider the variable $pack(w, i)$ equal to `true` if it is possible to fully pack a backpack of capacity w using the first i bars, and `false`, otherwise. The analysis of the two cases above leads to the following recurrence relation for $i > 0$,

$$pack(w, i) = pack(w, i-1) \text{ or } pack(w - w_{i-1}, i-1).$$